Proof that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

The proof follows Proof 11 of *Evaluating* $\zeta(2)$ at
`http://empslocal.ex.ac.uk/people/staff/rjchapma/etc/zeta2.pdf`

AB3 page 33 shows that if

$$I_n = \int_0^{\pi/2} \sin^n x \, dx$$

then

$$I_{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \cdots \cdot \frac{2n-1}{2n} \cdot \frac{\pi}{2}$$

It follows that

$$\begin{aligned} I_{2n} &= \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n-1) \cdot 2n}{(2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2n)^2} \cdot \frac{\pi}{2} \\ &= \frac{1 \cdot 2 \cdot 3 \cdot \cdots \cdot (2n-1) \cdot 2n}{(2^n \cdot 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n)^2} \cdot \frac{\pi}{2} \\ &= \frac{(2n)!}{4^n (n!)^2} \frac{\pi}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1) \end{aligned}$$

Substituting x by $\pi/2 - x$ shows that $I_n = \int_0^{\pi/2} \cos^n x \, dx$. Then use integration of parts to give

$$\begin{aligned} I_{2n} &= \left[x \cos^{2n} x\right]_0^{\pi/2} + 2n \int_0^{\pi/2} x \sin x \cos^{2n-1} x \, dx \\ &= n \left[x^2 \sin x \cos^{2n-1} x\right]_0^{\pi/2} - n \int_0^{\pi/2} x^2 \cos^{2n} x - (2n-1) \sin^2 x \cos^{2n-2} x \, dx \\ &= n(2n-1) J_{n-1} - 2n^2 J_n \end{aligned}$$

where $J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx$.

Hence

$$\frac{(2n)!}{4^n (n!)^2} \frac{\pi}{2} = n(2n-1) J_{n-1} - 2n^2 J_n$$

Multiply both sides by $\frac{4^n (n!)^2}{2n^2 (2n)!}$ to get

$$\begin{aligned} \frac{\pi}{4n^2} &= \frac{4^n (n!)^2}{2n^2 (2n)!} n(2n-1) J_{n-1} - \frac{4^n (n!)^2}{2n^2 (2n)!} 2n^2 J_n \\ &= \frac{4^n}{4(2n)!} \left(\frac{n!}{n}\right)^2 2n(2n-1) J_{n-1} - \frac{4^n (n!)^2}{(2n)!} J_n \\ &= \frac{4^{n-1} ((n-1)!)^2}{(2n-2)!} J_{n-1} - \frac{4^n (n!)^2}{(2n)!} J_n \end{aligned}$$

Now we use the telescoping series trick AA3 page 9 and get

$$\sum_{n=1}^{N} \frac{\pi}{4n^2} = J_0 - \frac{4^N (N!)^2}{(2N)!} J_N$$

We will show that

$$\lim_{N\to\infty} \frac{4^N (N!)^2}{(2N)!} J_N = 0$$

from which it will follow that

$$\sum_{n=1}^{\infty} \frac{\pi}{4n^2} = J_0 = \int_0^{\pi/2} x^2\, dx = \frac{\pi^3}{24}$$

so, using the multiple rule for series and multiplying both side by $4/\pi$ we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

as we wanted.

To show that

$$\lim_{N\to\infty} \frac{4^N (N!)^2}{(2N)!} J_N = 0$$

we use the *Inequalities for integrals* method of AB3 Section 3.1.

We also use the fact that $x \le \frac{\pi}{2}\sin x$ for $0 < x < \frac{\pi}{2}$ (proved later) with the Inequality Rule to get

$$\begin{aligned}
J_N &= \int_0^{\pi/2} N^2 \cos^{2N} x\, dx \\
&\le \int_0^{\pi/2} \left(\frac{\pi}{2}\sin x\right)^2 \cos^{2N} x\, dx \\
&= \frac{\pi^2}{4}\int_0^{\pi/2} \sin^2 x \cos^{2N} x\, dx \\
&= \frac{\pi^2}{4}\int_0^{\pi/2} (1-\cos^2 x)\cos^{2N} x\, dx \\
&= \frac{\pi^2}{4}\left(\int_0^{\pi/2} \cos^{2N} x\, dx - \int_0^{\pi/2} \cos^{2N+2} x\, dx\right) \\
&= \frac{\pi^2}{4}\left(I_{2N} - I_{2N+2}\right) \\
&= \frac{\pi^2}{4}\left(I_{2N} - \frac{2N+1}{2N+2} I_{2N}\right) \quad \text{by AB3 Exercise 2.3(b)} \\
&= \frac{\pi^2}{4}\frac{1}{2n+2} I_{2N}
\end{aligned}$$

So

$$\frac{4^N(N!)^2}{(2N)!}J_N \le \frac{4^N(N!)^2}{(2N)!}\frac{\pi^2}{4}\frac{1}{2n+2}I_{2N}$$
$$= \frac{\pi^2}{8(N+1)}\frac{4^N(N!)^2}{(2N)!}I_{2N}$$
$$= \frac{\pi^2}{8(N+1)}\frac{\pi}{2} \quad \text{by (1)}$$
$$= \frac{\pi^3}{16(N+1)}$$

Thus

$$0 < J_N \le \frac{\pi^3}{16}\frac{1}{n+1}$$

By the Squeeze Rule for sequences it follows that

$$\lim_{N\to\infty} \frac{4^N(N!)^2}{(2N)!}J_N = 0$$

and we are done.

Finally we must show

$$x \le \frac{\pi}{2}\sin x \text{ for } 0 < x < \frac{\pi}{2}$$

which we can do using Strategy 4.1 of AB2. Let

$$f(x) = \frac{x}{\sin x} \text{ for } 0 < x < \frac{\pi}{2}.$$

From now on assume $0 < x < \frac{\pi}{2}$

$$f'(x) = \frac{\sin x - x\cos x}{x^2} \ge 0 \text{ as } \cos x \le \frac{\sin x}{x} \text{ by AB1 page 7}$$

so f is increasing on $(0, \frac{\pi}{2})$ and hence, on this interval,

$$f(x) \le f\left(\frac{\pi}{2}\right) = \frac{\pi/2}{\sin \pi/2} = \frac{\pi}{2}.$$

Thus

$$\frac{x}{\sin x} \le \frac{\pi}{2}$$

so

$$x \le \frac{\pi}{2}\sin x.$$