Theorem *If g is an element of $G = S_n$ then $C_G(g) = \langle g \rangle$ if and only if the cycles of g are of unequal coprime lengths.*
Alternatively, $C_G(g) = \langle g \rangle$ if and only if g has cycles of coprime length with at most one 1-cycle.

Proof.
Since powers of g centralise g it follows that $\langle g \rangle \subseteq C_G(g)$ so

$$C_G(g) = \langle g \rangle \iff |\langle g \rangle| = |C_G(g)| \iff |g| = |C_G(g)| \tag{1}$$

Let G act on itself by conjugation. Then $xgx^{-1} = x \iff xg = gx$ so

$$\text{Stab}(g) = C_G(g) \tag{2}$$

$$\begin{aligned} |\text{Orb}(g)| &= \text{the number of distinct conjugates of g} \\ &= \text{the number of permutations with the same cycle structure as } g \end{aligned}$$

Let a be the product of the lengths of the cycles of g
Let b be the number of cycles of equal length.
Let l be the least common multiple of the lengths of the cycles, $|g| = l \leq a$.
Then the number of permutations with the same cycle structure as g is

$$\frac{n!}{ab} = \frac{|G|}{ab}$$

Hence

$$|\text{Orb}(g)| = \frac{|G|}{ab} \tag{3}$$

By the Orbit-Stabiliser theorem, $|\text{Orb}(g)| \times |\text{Stab}(g)| = |G|$ so by (3)

$$|\text{Stab}(g)| = \frac{|G|}{|\text{Orb}(g)|} = \frac{|G|}{|G|/ab} = ab \geq |g|b \geq |g| = l$$

and it follows that

$$|\text{Stab}(g)| = |g| \iff a = l \text{ and } b = 1 \tag{4}$$

But

$$a = l \iff g \text{ has coprime cycle lengths} \tag{5}$$
$$b = 1 \iff g \text{ has unequal cycle lengths} \tag{6}$$

The result now follows from (1), (2), (4), (5) and (6).

$$C_G(g) = \langle g \rangle \iff |\text{Stab}(g)| = |g| \iff \text{the cycles of } g \text{ are of unequal coprime lengths}$$

Cycles of coprime length will have unequal lengths unless they are 1-cycles. Hence we may replace *unequal* by *at most one 1-cycle.*

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