Introduction

Many people interested in education would have seen headlines like Maths 'stuck in downward spiral' quoting from a report from the UK Mathematics Foundation, with quotes such as

> *Changes brought in in England in 2000, which divided A-levels into two separate parts - divided into modules - had been the "most recent and most public nail in the coffin" of decline.*
>
> *They had made it "impossible to teach and to assess mathematics in an integrated way", making the subject "less appetising".*

So what's going on? How did we get here? How should A level maths be taught? This document is an attempt to answer these questions.

The biggest problem that the teaching of mathematics faces is ignorance. Those involved in education but who are not mathematicians have finally come to realise how hard the subject is, with many reports stating that it is the hardest A level subject of all. Despite this, the teaching of mathematics is influenced by those that have little feel for the subject and who impose their own systems, then complain about the consequences.

For example, how many non-mathematicians believe that

- maths problems have only one solution;
- answers can only be right or wrong;
- mathematics rests on solid logical grounds;
- A level should teach correct mathematics and
- maths is a science?

Each one of these is a misconception and this document will try to clear this up.

If you, dear reader, know[1] that the statements

$$1+1=2$$
$$1+1=0$$
$$1+1=10$$

are, in the right context, all true statements, then you are to be congratulated; your knowledge of mathematics is beyond that of the general public ☺[2].

[1] 'Know' means being able to explain why they are correct.

[2] I am acutely aware that this is partly the fault of mathematicians not explaining their subject well enough to others (hence this document). It is a difficult subject to explain, so I am a great fan of those like Martin Gardner, Ian Stewart and Keith Devlin who do it so well. *Everyone* interested in mathematics should read their books and articles.

History

The history of mathematics itself is an absolutely fascinating subject which I try to bring into my teaching wherever possible[3]. But here we will concentrate on the relevance of this history to A level maths.

In 1988 GCSE replaced O level. O level was taken by the top 20% of students, whereas GCSE aims to pass the top 50%. This attempt to 'democratise' the subject can only be done in mathematics by making the syllabus less demanding. There is nothing unusual in this; it has been going on for at least the last 100 years. If one looks at equivalent papers of the early 20$^{\text{th}}$ century, some degree students would struggle to answer them.

As a consequence, for the next 10 years or so, the A level mathematics syllabus had to be made easier to keep the (still large) gap between it and GCSE manageable. This then had a knock-on effect in the Universities where they found that students were arriving with a less comprehensive knowledge of mathematics and so struggled in subjects like Engineering, Sciences, Economics etc as well as Mathematics. They attempted to overcome this problem with extra introductory modules as well as lengthening some courses to 4 years.

Nevertheless, this was just a stop-gap approach and, as a result, in 1995 the London Mathematics Society report entitled "Tackling the Mathematics Problem" was published. This report received scant publicity generally, but its influence cannot be underestimated and should be compulsory reading for anyone interested in mathematics teaching[4].

It says some pretty devastating things about the state of mathematics at the time:

> *Recent changes in school mathematics may well have had advantages for some pupils, but they have not laid the necessary foundations to maintain the quantity and quality of mathematically competent school leavers and have greatly disadvantaged those who need to continue their mathematical training beyond school level.*
>
> *A major cause of these problems has been the flawed method of planning change in the past decade. There is no representative, authoritative, continuing forum for mathematics, bringing together mathematicians, scientists, engineers, employers, teachers etc. Rather, there is a one-sided dialogue between SCAA and individual bodies, with agenda-setting and decision-making controlled by a small and necessarily unrepresentative group within SCAA.*

[3] I am always delighted to discover that Norwegian students not only know who Abel was but why he is so famous. Few British students can say anything about Newton ☹.

[4] I recommend downloading and printing out the postscript version, rather than relying on the HTML version. I can also provide a pdf version.

> *The situation was exacerbated by the revised compulsory A-level core produced by SEAC/SCAA which appears to attempt more to accommodate diverse types of courses than to provide the predictable base needed by end-users. As will be seen from Appendix B, the amount of material that is now common to all the boards has been reduced to the point where those in higher education can infer relatively little from the fact that a student has a 'mathematics' A-level. The kind of differences revealed in that table not only make life very difficult for those in higher education, but (as far as one can tell) bring no obvious benefits to schools and colleges.*
>
> *The report DfE (1994) highlighted inadequacies in the flow of qualified young people into science and technology. We welcome this official recognition of some of the problems and support much that is to be found therein. However, the picture presented by it, and by the subsequent report OFSTED (1994), is too often obscured by a general air of complacency. For example, both reports take comfort in the fact that graduate numbers in mathematics and computer science combined have expanded steadily, even though the picture for mathematics alone is more worrying. All our evidence suggests (see Sections 5–10) that such statements as "The picture, therefore, can be said to be relatively reassuring" (OFTSED[5], 1994, p.1) are dangerously misleading.*

It was a well-written and sensible report[6], but it then triggered some unintended and unwelcome consequences. The report was in the forefront of the minds of those writing syllabuses for Curriculum 2000. As a result, the A level mathematics syllabus was strengthened with many topics, which had been dropped in the 90s, put back in. The result was a national crisis in 2001[7], reported in detail in the press, with large numbers of AS mathematics students failing and not going on to A2.

Estelle Morris, the Education Minister, instigated an inquiry which led to yet another large syllabus change, which we are half-way through, as well as interim measures such as an extra examination session in November. The situation can only be described as chaotic, with the panic resulting in no textbooks being published for the A2 syllabus (a good textbook is at the heart of A level teaching) and a new dramatically overloaded AS syllabus.

The Universities were also dismayed to find that many students were no longer applying to study for mathematics degrees with the result that many mathematics departments have had to close[8]. Indeed, there are now only around 7000 students studying mathematics in the whole of the UK. [redacted] University has readjusted all of its courses to include as little mathematics as possible to make them more attractive to students.

The situation is as desperate as ever. This is, in my opinion, a result of the lack of

[5] *sic*

[6] Of course, since it was written by mathematicians ☺.

[7] Not in [redacted] though. Our results actually improved and were way above the national average. I like to think it's due to the excellent teaching our students received from experienced mathematicians. Indeed, the A2 results in 2002 were absolutely stunning.

[8] This is an excellent example of the *Law of Unintended Consequences.*. Universities were so worried by the mathematical ability of their students that they encourage changes which results in closures of mathematics departments.

coordination between the various levels of mathematics teaching. Mathematics is very much a linear subject and altering one part of it without reference to years above and below cannot work. Many years ago, in a newspaper survey, I asked readers to write in telling me why they found mathematics so hard. Their overwhelming response was that if they missed lessons for whatever reasons, it was very difficult to catch up later.

It is essential that if reforms are made they start in the primary schools and then propagate upwards through secondary schools and FE to universities. The problem is that this, of necessity, can take up to 20 years to achieve. Is the introduction of the numeracy hour in primacy schools the start of a long-term solution?

> *We are facing serious challenges. Only if all parts of the education system work together will true progress be made.* Tackling the Mathematics Problem

The Subject

In order to teach/inspect/observe mathematics well it is essential that the subject is properly understood. It is a well-known rule of thumb that mathematics teachers need at least one higher qualification in mathematics than they teach. Thus A level maths teachers should have a degree in maths (and not just in a related subject) and the same goes for inspectors and observers (which has been true for FEFC & OFSTED inspectors in my classes).

But this doesn't happen everywhere due to the shortage of suitably qualified teachers and with those involved in teacher training who should know better. So for their benefit here are a few answers to questions about mathematics that need to be understood by those looking at A level mathematics teaching.

What is mathematics?

How many people when asked this question can answer it with any degree of understanding? The best answer that I have seen is that it is the **Study of Pattern**. See AIMS Plans Pattern-Based Math/Science Curriculum for example, but mainly I leave it as an *important* exercise for the reader to find out why this is a good definition.

Is Mathematics an Art or a Science?

Most people believe this to be a very simple question. After all, much of mathematics was invented to solve scientific problems (Newton invented differential calculus to show that planets orbit the sun in an ellipse). In fact mathematics is sometimes known as the Queen of Science.

Yet, this is a far too simplistic approach. Much of mathematics is invented[9] out of pure interest in the subject – it may happen that years later it *does* solve problems. Indeed, that is what happens in the vast majority of cases and history is littered with examples – Fermat is a famous such example. Mathematicians talk of beauty

[9] The question whether mathematics is invented or discovered is an important one but, sadly, not covered here.

and approach their subject in similar ways that others write poetry[10]. It is this approach that puts mathematics in the Arts camp. Warwick University allows its maths students to choose between a BA and BSc when they graduate.

Why is this important? Because it shows that non-mathematicians have a view of the subject that is not correct, so cannot be expected to make useful judgements of the way it is taught.

How can 1 + 1 not be 2?

It all depends on context. In so-called clock arithmetic, which is based on the number 12, 4 hours after 10 o'clock is 2 o'clock, so one can write $10 + 4 = 2$. This is an example of
modular arithmetic where one writes the result of the addition as the remainder after division by 12.

If instead of 12 one uses 2 you get $1 + 1 = 0$.

Binary arithmetic, vital to computers, says $1 + 1 = 10$ where 10 means $1 \cdot 2 + 0$ in normal (denary) arithmetic.

Shouldn't correct mathematics be taught at A level?

Definitely not! Mathematics is so complicated and difficult that teaching everything absolutely correctly would be well beyond even the best students' capabilities. For example, differential equations are taught so that the method only works for the simplest of functions. Once this is *fully* understood the degree student can see the faulty method and how it needs to be repaired to apply to all cases.

This is very common at all levels of mathematics. First, it is taught in a rather simplistic way and only later is this overtaken by a more rigorous approach. In the same way, one learns that $1 + 1 = 2$ and only later does one realise that this is a rather naïve view.

Isn't mathematics is a totally logical subject?

This is a very important question. Anybody involved in A level teaching, and beyond, needs to understand the quicksand on which the 'tower' of mathematics is built.

In the 19$^{\text{th}}$ century, mathematicians got rather worried about the paradoxes[11] that seemed to appear in the subject. They were aware that any (and I mean *any*) inconsistency in mathematics undermines the whole subject and much of our civilisation[12]. So in 1900 David Hilbert proposed that mathematics should be put on a

[10] See for example It adds up to beauty. The equation $e^{i\pi} = -1$ is famous as being considered the most beautiful equation. If you don't truly understand why then should you be looking at A level maths? See Mathematical Formulae.

[11] It is not difficult to show that $1 + 1 - 1 + 1 - 1 + \ldots$ can be made to equal any real number whatsoever.

[12] This is *not* a joke. It is well-known that $A \& \neg A \Rightarrow B$ is true ie if a statement is both true and false then *every* statement is true. Bertrand Russell gave a convincing proof, using a contradiction, that he was the Pope.

rigorous basis, with mechanical procedures for proving everything from the axioms.

In 1931 there was a revolution in mathematics similar to the one in Physics triggered by Relativity and Quantum Theories. In that year Gödel proved that the axiomatic approach was not capable of proving all true theorems. There were some true results that could never be proved, and furthermore, we could not know which these unprovable results were. These ideas were taken up by others such as Turing who cracked the Enigma code and built the first electronic computer; he went on to look at the limits of what computers, however powerful, can do.

It meant in essence that mathematics would remain forever on a dodgy logical basis, but, on the other hand, is not a mechanical subject, nor could it be done by a computer. As Gödel himself put it:

> *Either mathematics is too big for the human mind or the human mind is more than a machine.*

This contradiction between the mechanical approach required for A level and the true nature of mathematics needs to be understood by those teaching the subject.

Day to day teaching

Despite my wish to impart a love of the subject, the primary objective of AS & A2 classes is to get students through the exams. The problem is that the syllabus is very heavily loaded and the exams themselves can be as early as May.

On top of that, the difficulty of the subject means that it can take a year for the student to understand a topic. Given the linearity of mathematics and the fact that the exams are 8 months after the course starts, this is a severe problem.

An approach that has proved successful over at least the last 5 years, is to go through the syllabus very quickly indeed with the aim if finishing it by mid-March. The remaining 2 months are spent in revision, going through as many past papers as possible[13]. This brutal approach works because it is only at the end of the course with constant repetition does it all start to fall into place.

It does mean that not a minute can be wasted in class. There can be no discussion nor little use of computers in the class itself .

Indeed, in a new "back to basics" approach, the exam boards discourage the use of calculators and other such aids unless absolutely necessary. One of the modules bars calculators altogether so students need to learn how to do without them which can means re-learning arithmetic they had forgotten.

A level mathematics is about teaching mechanical approaches[14]. As Tackling the Mathematics Problem complains:

[13] and often *more* than is possible.

[14] In the 60s a new approach (called Modern Maths) tried to get students to understand mathematics as a way of learning it. It was a disastrous failure with students learning very little and understanding less. It was abandoned with recriminations flying about and is unlikely to arise again in the foreseeable future.

> *Students enrolling on courses making heavy mathematical demands are hampered by a serious lack of* essential technical facility *– in particular, a lack of fluency and reliability in numerical and algebraic manipulation and simplification.*

They also say:

> *During the same period we have also seen implicit 'advice' (from HMI (1985), from OFSTED(1994), in the wording of the National Curriculum, and from elsewhere) that teachers should reduce their emphasis on, and expectations concerning, technical fluency. This trend has often been explicitly linked to the assertion that "process is at least as important as technique". Such advice has too often failed to recognise that to gain a genuine understanding of any process it is necessary first to achieve a robust technical fluency with the relevant content. Progress in mastering mathematics depends on reducing familiar laborious processes to automatic mental routines, which no longer require conscious thought; this then creates mental space to allow the learner to concentrate on new, unfamiliar ideas (as one sees, for example, in the progression from arithmetic, through fractions and algebra, to calculus).*

Much like young children are taught multiplication tables by repetition, advanced techniques are taught by the use of exercises. Students need to be taught a technique and then practise it by doing as many exercises as time allows and they can stand. Thus classes include an explanation on the board (which can require vast amounts of space – only a large roller board is suitable) followed by examples followed by exercises. Yes, one tries to lighten things up by historical references or jokes, but there is no time for anything else. **It is incumbent on an observer to understand this**. **There is no excuse for those who don't.** FEFC & OFSTED inspectors expect and insist that this happens.

The need to be efficient and not waste a minute means that I always impose the following rule:

In my mathematics classes, everyone is either a student or a teacher. There is no other possibility. Thus if an outsider is not willing to help the teaching[15] then they are students and must take notes and do the exercises.

This has the benefit of helping students to understand the seriousness of my approach.

This is far from the sort of approach I would like to use but is essential given the pressures put on both staff and students to succeed[16].

[15] with which I am pleased to say inspectors are always happy to oblige.

[16] How can we expect students to learn to love the subject in the way I do if we don't have the chance to broaden the curriculum? UK Mathematics Foundation feels the same and this underlies the sadness of their report.

Appendix

This document can be downloaded from http://sixthform.info/maths/files/AMaths2.pdf and then the links become live.

For those reading a printed version the links are also given here:

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	report	http://image.guardian.co.uk/sys-files/Education/documents/2005/06/28/maths.pdf
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	modular arithmetic	http://en.wikipedia.org/wiki/Modular_arithmetic
	It adds up to beauty	http://www.guardian.co.uk/saturday_review/story/0,3605,639232,00.html
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